Uncertainty Analysis

WMO TRAINING WORKSHOP ON METROLOGY FOR SOUTHWEST PACIFIC RA V ENGLISH SPEAKING COUNTRIES

Melbourne, Australia, 21-25 November 2011
What To Expect

• The subject is taught to Bureau staff by Australia’s National Measurement Institute over 3 days!
• At the end of this session you will be able to:
  • Understand the need for uncertainty analyses compliant with the ISO-GUM.
  • Read an uncertainty analysis and understand it.
  • Understand the process used to produce an Uc analysis.
  • Produce simple uncertainty analyses.
ISO-GUM

• The International Standards Organization Guide to Uncertainty in Measurement.
• It is a guide – conventions are developing as laboratories develop more experience.
• It attempts to standardise practices for reporting uncertainties.
I am a Barometer

- In pairs roll 3 dice each
- Add up the dice
- Average the dice for each pair
- So what is the mean pressure?
- What is the expected pressure?
- A 95 cent bet
Predictability

- Bias – the distance between the average value and the true value
- Repeatability – the variation around the mean
- Uncertainty
Basic ISO Tenets

- The ISO approach is based on the following rules:
- Each uncertainty component is quantified by a standard deviation.
- All biases are assumed to be corrected and any uncertainty is the uncertainty of the correction.
- Zero corrections are allowed if the bias cannot be corrected and an uncertainty is assessed.
- All uncertainty intervals are symmetric.
Uncertainty

• Parameter, associated with the result of a measurement, that characterises the dispersion of the values that could reasonably be attributed to the measurand. (International Vocabulary of Basic and General Terms of Metrology)

• The degree of doubt about a measurement! You will always be correct in your measurement if you make your uncertainty estimate large enough!

• However; if the uncertainty becomes too large the measurement loses its value.
Why Bother?

• It is a requirement for most accreditation schemes (17025, national testing associations etc)
• It allows laboratories to reliably compare results.
• It allows your laboratory to identify the significant contributors to the measurement errors within your lab.
Example Inter-Comparison

- Spectral Radiometry – BIPM
- Note the error bars (95% confidence) are saying ‘we believe the true value lies between these points’
Inputs to the Uncertainty Analysis

- Components are grouped into two major categories, depending on the source of the data and not on the type of error, and each component is quantified by a standard deviation. The categories are:
  - Type A - components evaluated by statistical methods
  - Type B - components evaluated by other means (or in other laboratories)
The Process

- **Reported value involves measurements on one quantity.**
- Compute a type A standard deviation for random sources of error from:
  - Replicated results for the test item.
  - Measurements on a check standard.
- Make sure that the collected data and analysis cover all sources of random error such as:
  - Instrument imprecision
  - Day-to-day variation
  - Long-term variation
  - Resolution
  - Uncertainty of reference
- and bias such as:
  - Differences among instruments
  - Operator differences.
- Compute a standard deviation for each type B component of uncertainty.
- Combine type A and type B standard deviations into a standard uncertainty for the reported result using sensitivity factors.
- Compute an expanded uncertainty.
Standard Uncertainty

- **Standard Uncertainty**: Represent each component of uncertainty that contributes to the uncertainty of the measurement result by an estimated standard deviation $u_i$
- Termed **standard uncertainty**, equal to the positive square root of the estimated variance $u_i^2$.
- See plots opposite. Estimate the std dev. $S_i$.
- **Mean** = -1.51, $s = 2.22$
Types of Distributions

• Not everything has a normal distribution.
• Sometimes the distribution of a parameter is unknown and can only be guessed.
• Common distribution types are:
  • Gaussian – signal noise
  • Rectangular – for example water in a rain gauge
  • Triangular – used if you are sure of the end points and believe the mode occurs at zero.
Rectangular Distribution

- This is the most conservative estimate of uncertainty since it leads to the largest value of $s$.
- An actual example if the water in the siphon of a tipping bucket rain gauge at an particular time.

\[ s_{\text{source}} = \frac{1}{\sqrt{3}} a \]
Triangular Distribution

- The triangular distribution leads to a less conservative estimate of uncertainty; i.e., it gives a smaller standard deviation than the uniform distribution.
- The calculation of the standard deviation is based on the assumption that the end-points, ± a, of the distribution are known and the mode of the triangular distribution occurs at zero.

\[ s_{\text{source}} = \frac{1}{\sqrt{6}} a \]
• The normal distribution leads to the least conservative estimate of uncertainty; i.e., it gives the smallest standard deviation.

• The calculation of the standard deviation is based on the assumption that the end-points, $\pm a$, encompass 99.7 percent of the distribution.
Combining Uncertainties

- All uncertainty components (standard deviations) are combined by root-sum-squares to arrive at a 'standard uncertainty', $u$, which is the standard deviation of the reported value, taking into account all sources of error, both random and systematic, that affect the measurement result.
- Do not make the mistake of 'adding errors' – this is the old approach.
- Adding errors (i.e. adding $s$, rather than $s^2$) is equivalent to saying all of the sources of error are exactly correlated.
Creating a Measurement Model

• You need a model of the measurement in order to apply the mathematics.
• It’s not as terrifying as it sounds – the model can be quite simple.
• For example, if you were comparing a reference thermometer to a field thermometer then the model might be:
  • \( C = \text{Reference Temperature} - \text{Field Device Temperature} \)
• Note; \( C \) is the measurand, in this case the correction to the field thermometer’s temperature – not temperature.
• Setting up the model is the critical step – if the model is wrong the uncertainty analysis will be wrong.
• The simplest method for setting up the model is to follow the flow of information about the measurement.
Simple Measurement Model

- Calculate the uncertainty of an area;
- \( area = length \times width \)
- The formal propagation of error approach is to compute:
  - standard deviation from the length measurements
  - standard deviation from the width measurements
- and combine the two into a standard deviation for area using the approximation for products of two variables (ignoring a possible covariance between length and width)

\[
s_{area} = \sqrt{width^2 s_{length}^2 + length^2 s_{width}^2}
\]
The NMI3 method for calculating the pressure set by a piston gauge is given by equation (1). A description of each of the input parameters is covered in depth by the NMI3 and is not repeated here.

And the symbol definitions and units are given by:

- \( P \) Pressure under piston \( \text{Pa} \)
- \( P' \) Estimate of \( P \) \( \text{Pa} \)
- \( P_{\text{vac}} \) Pressure of the vacuum above the piston \( \text{Pa} \)
- \( \rho_a \) Density of the air above the piston \( \text{kg.m}^{-3} \)
- \( \rho_{\text{Mi}} \) Density of the individual load mass \( \text{kg.m}^{-3} \)
- \( \rho_p \) Density of air below the piston \( \text{kg.m}^{-3} \)
- \( g \) Local gravity \( \text{m.s}^{-2} \)
- \( M_i \) Mass of the individual load masses \( \text{kg} \)
- \( M_e \) Effective mass of the piston \( \text{kg} \)
- \( A_{0,20} \) The effective cross-section of the piston at 20 °C \( \text{m}^2 \)
- \( \lambda \) The pressure distortion coefficient \( \text{Pa}^{-1} \)
- \( \alpha_P \) Linear thermal expansion coefficient of the piston °C\(^{-1} \)
- \( \alpha_C \) Linear thermal expansion coefficient of cylinder °C\(^{-1} \)
- \( T \) Temperature of the piston cylinder °C
- \( h \) Height of reference point of barometer m

\[
P = \frac{\sum_{n} M_i g (1 - \frac{P_a}{\rho_{M_i}}) + M_e g}{A_{0,20} (1 + \lambda P') \left[1 + (\alpha_P + \alpha_C)(T - 20)\right]} + P_{\text{vac}} - h \rho_p g
\]
Typical Uncertainty Inputs

- Uncertainty of reference device used.
- The error associated with repeat measurements.
- The error resulting from the finite resolution of the measuring device and/or quantity being measured.
- The error introduced by variations in environmental conditions or by correcting for environmental conditions.
- The error introduced by digitizing an analog signal.
- The error introduced by the person making the measurements……..etc.
Type A Analysis

- Type A inputs are calculated from experimental data.
- Typically, a data stream is used to calculate a sample mean and standard deviation ($x, s$).
- The type of distribution may have to be assumed.
Type B Analysis

- A Type B evaluation of standard uncertainty is usually based on scientific judgment
- previous measurement data,
- experience of the behavior of instruments
- manufacturer's specifications
- data provided in calibration and other reports
- uncertainties assigned to reference data taken from handbooks.
- For a type B analysis the underlying distribution of experimental error is estimated.
- It is necessary to estimate the type of distribution
- A typical Type B input is the Uc from a calibration certificate.
Estimating $\sigma$

- For sparse data (< 50) the calculated std deviation may not be the best estimator of the population variance.
- Often the ‘semi-range’ is employed.
- That is; the highest sample value minus the lowest sample value is assumed to equal $4 \sigma$. 

![Graph showing sample numbers versus error]
Degrees of Freedom

• The calculation of the degrees of freedom drives the uncertainty analysis in terms of confidence.
• The greater the degrees of freedom the higher your confidence in the parameter.
• The degrees of freedom are used to calculate the Coverage Factor.
Type B Degrees of Freedom

- When the distribution parameters have been estimated you have the problem of estimating the degrees of freedom.
- Most workers estimate the degrees of freedom using the following guide.
  - $V = 10$  Low confidence
  - $V = 50$  High confidence
  - $V = 100$  Very high confidence
- These values of $v$ are input into the Uc analysis
Combining Degrees of Freedom

- Degrees of freedom are combined using the Welch-Satterwaite formula shown opposite.
- Note the formula contains the calculated uncertainties $U_i$ and the combined uncertainty $U_c$.
- The formula has the effect of weighting the various degrees of freedom based on their components contribution to the combined uncertainty.
Sensitivity Analysis

- The mathematical model is examined to determine how sensitive the measurand is to variations in other parameters.
- The sensitivity coefficient shows the relationship of the individual uncertainty component to the standard deviation of the reported value for a test item.
- The sensitivity coefficient relates to the result that is being reported and not to the method of estimating uncertainty components where the uncertainty, \( u \), is

\[
u = \sqrt{\sum_{i=1}^{R} \alpha_i^2 s_i^2}
\]

- For example: suppose \( C = M_{\text{ref}} - M_{gh} \)
- This involves taking the partial derivative of the model with respect to the parameter of interest.
Numeric Sensitivity Analysis

- Numerical sensitivity analysis is easier and in some cases the only way to proceed due to the intractability of the mathematical model.
- It is done by making small variations in parameters and noting the effect on the measurand.
- Easily done in a spreadsheet.
Correlations

- So far it has been assumed that one parameter in the uncertainty analysis does no influence any other measured parameters.
- A typical example is the weighing of an object using fixed masses.
- There are standard techniques for handling correlations.
What’s in – What’s not?

- Generally a lab will account for any systematic errors; that is, the known offsets or biases in their instruments.
- If they are not known, or not accounted for, then they must be included in the uncertainty analysis.
- If they have been accounted for then they are not included.
- For example: in the top graph the uncertainty in voltage could be calculated between the blue lines if the drift was allowed for.
- In the second graph the uncertainty of the RTDs is quite low if their linearity is accounted for.
- If it isn’t then the uncertainty for any RTD is $0.12/4$ ie. The expected variability of the devices.
The Process

• Step 1 – Model the measurement
• Step 2 – List all error sources
• Step 3 – Characterise all error components
• Step 4 – Get components of standard uncertainty for measurand
• Step 5 – Calculate the combined standard uncertainty
• Step 6 – Calculate the expanded uncertainty
• Step 7 – State the result
The optical density (o.d.) of a fluid is given by:

\[ \text{o.d.} = \varepsilon C l \]  

(4.10)

\( \varepsilon \) is the extinction coefficient, \( C \) is the concentration coefficient of the absorbing species in the liquid, and \( l \) is the path length of the light. The best estimate of each quantity, standard uncertainty and degrees of freedom are as follows:

\( \varepsilon = 14.9 \text{ L mol}^{-1} \text{ mm}^{-1}; \ u(\varepsilon) = 1.2 \text{ L mol}^{-1} \text{ mm}^{-1}; \nu_{\varepsilon} = 5 \)

\( C = 0.042 \text{ mol L}^{-1}; \ u(C) = 0.003 \text{ mol L}^{-1}; \nu_{C} = 7 \)

\( l = 1.42 \text{ mm}; \ u(l) = 0.21 \text{ mm}; \nu_{l} = 8 \)

Determine,

a) the best estimate of the optical density, o.d.
b) the combined standard uncertainty
c) the effective degrees of freedom, \( \nu_{\text{eff}} \)
d) the coverage factor \( \nu_{\text{eff}} \text{ calculated in part c)} \) (assume that the level of confidence is 0.95).
The Process

- Step 1 – Model the measurement
- Step 2 – List all error sources
- Step 3 – Characterise all error components
- Step 4 – Get components of standard uncertainty for measurand
- Step 5 – Calculate the combined standard uncertainty
- Step 6 – Calculate the expanded uncertainty
- Step 7 – State the result
Using equation 4.2, this gives,

\[ u_c^2(y) = u_1^2(y) + u_2^2(y) + u_3^2(y) = 0.05122 + 0.004029 + 0.01727 = 0.02642 \]

It follows that \( u_c = 0.1625 \)

c) To find the effective degrees of freedom, \( v_{eff} \), we use equation 4.9, such that,

\[ v_{eff} = \frac{u_c^4(y)}{\sum_{i=1}^{N} \frac{u_i^4(y)}{v_i}} = \frac{(0.02642)^2}{(0.05122)^2 + (0.004029)^2 + (0.01727)^2} = 15.56 \text{ (round to 15)} \]

d) When the number of degrees of freedom is 15, the coverage factor for a level of confidence of 0.95 is 2.131 (see table 2 in appendix 1 of Kirkup (2002) for table of critical values for the \( t \) distribution)
Part of the hand-out cd is an Excel spreadsheet developed by the CSIRO. It allows the rapid calculation of 95% confidence limits. It is only as good as the information you put into it. Another method is @Risk. This is an add-on package for Excel and simulates probability distributions.
Table 1. – $F_s/F_c$ uncertainty analysis of typical Berlin Athlete data for a Reference Normal spring.

<table>
<thead>
<tr>
<th>Component</th>
<th>Units</th>
<th>Distribution</th>
<th>U or a</th>
<th>Coverage k</th>
<th>df</th>
<th>u(l)</th>
<th>c(l)</th>
<th>u(l)c(l)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spring</td>
<td>%</td>
<td>normal</td>
<td>0.76</td>
<td>2.31</td>
<td>8</td>
<td>0.3290043</td>
<td>1</td>
<td>0.3290043</td>
</tr>
<tr>
<td>mass</td>
<td>%</td>
<td>normal</td>
<td>0.25</td>
<td>2</td>
<td>30</td>
<td>0.125</td>
<td>0.5</td>
<td>0.0625</td>
</tr>
<tr>
<td>drop height</td>
<td>%</td>
<td>normal</td>
<td>0.45</td>
<td>2</td>
<td>30</td>
<td>0.225</td>
<td>0.5</td>
<td>0.1125</td>
</tr>
<tr>
<td>conc. resol.</td>
<td>%</td>
<td>Rect</td>
<td>0.05</td>
<td>1.732</td>
<td>100000</td>
<td>0.0288684</td>
<td>1</td>
<td>0.0288684</td>
</tr>
<tr>
<td>conc. meas.</td>
<td>%</td>
<td>normal</td>
<td>1</td>
<td>2.78</td>
<td>4</td>
<td>0.3597122</td>
<td>1</td>
<td>0.3597122</td>
</tr>
<tr>
<td>samp. resol.</td>
<td>%</td>
<td>Rect</td>
<td>0.05</td>
<td>1.732</td>
<td>100000</td>
<td>0.0288684</td>
<td>1</td>
<td>0.0288684</td>
</tr>
<tr>
<td>samp. meas.</td>
<td>%</td>
<td>normal</td>
<td>1</td>
<td>2.78</td>
<td>4</td>
<td>0.3597122</td>
<td>1</td>
<td>0.3597122</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>FR</th>
<th>Uncertainty</th>
<th>Sums combined s.d.</th>
<th>Effective df</th>
<th>k</th>
<th>Expanded uncertainty</th>
<th>rounded</th>
<th>Uncertainty in FR</th>
</tr>
</thead>
<tbody>
<tr>
<td>55</td>
<td>0.68</td>
<td>0.6206923</td>
<td>15.081178</td>
<td>2</td>
<td>1.3158676</td>
<td>1.32</td>
<td>0.92</td>
</tr>
<tr>
<td>40</td>
<td>0.85</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>0.89</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>0.92</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The Spread Sheet

- The data is input into the spreadsheet
- The type of distribution
- The units

Do type B calculations here for $u_i, c_i, v_i$.

Expand table if required and insert values.

<table>
<thead>
<tr>
<th>Uncertainty Component</th>
<th>Units</th>
<th>Distribution Type</th>
<th>Evaluation Type</th>
<th>Range U or a</th>
<th>Divisor d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>mV</td>
<td>Normal</td>
<td>A</td>
<td>1.00E-03</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>A</td>
<td>Rectang.</td>
<td>B</td>
<td>3.00E-03</td>
<td>0.577350269</td>
</tr>
<tr>
<td>3</td>
<td>T</td>
<td>N</td>
<td>A</td>
<td>5.60E-03</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>g</td>
<td>R</td>
<td>B</td>
<td>3.00E-03</td>
<td>0.40824829</td>
</tr>
</tbody>
</table>
- Sensitivity factors come from the model.
- Degrees of freedom are calculated – \( n-1 \) for Type A, or estimated – Type B.

\[
S = \frac{V}{\sigma \psi (T_e^4 - T_s^4) + \sigma k (T_d^4 - T_e^4)}
\]

<table>
<thead>
<tr>
<th>Degrees of Freedom</th>
<th>Standard. Uncertainty (( u_i ))</th>
<th>Sensitivity factor (( c_i ))</th>
<th>( c_i u_i )</th>
<th>(( c_i u_i ))^2</th>
<th>(( c_i u_i ))^4/( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>0.001</td>
<td>1</td>
<td>0.001</td>
<td>0.000001</td>
<td>5.55556E-14</td>
</tr>
<tr>
<td>8</td>
<td>0.051961524</td>
<td>16</td>
<td>0.831384388</td>
<td>0.6912</td>
<td>0.05971966</td>
</tr>
<tr>
<td>19</td>
<td>0.0056</td>
<td>0.23</td>
<td>0.001288</td>
<td>1.65884E-06</td>
<td>1.44847E-13</td>
</tr>
<tr>
<td>100</td>
<td>0.007346469</td>
<td>5.1</td>
<td>0.00140454</td>
<td>0.00140454</td>
<td>1.97273E-06</td>
</tr>
</tbody>
</table>

**SUMS**             | **\(0.692607199 \)**           | **0.0597197**               |

**Combined uncertainty** \(0.832230256\)

**Effective degrees of freedom** \(3.032604555\)

**Coverage factor** \(2.306004133\)

**Expanded Uncertainty** \(1.91912641\)
When reporting a measurement result and its uncertainty, include the following information in the report itself or by referring to a published document:

- A list of all components of standard uncertainty
- Their degrees of freedom
- The resulting value of $U_c$
- The components should be identified according to the method used to estimate their numerical values:
  - A. those which are evaluated by statistical methods
  - B. those which are evaluated by other means.
- A detailed description of how each component of standard uncertainty was evaluated.
- A description of how $k$ was calculated
Using an Uc Analysis

- An uncertainty analysis can help you determine the factors that are making your measurements less certain (less accurate).
- Often as in the case below, one or two items are contributing the majority of the uncertainty.

![Bureau's Piston Gauge Uncertainty Contributors](image)
• Note: in the previous example the largest contributor to $U_c$ is the pressure reading from the vacuum gauge.

• The $U_c$ of pressure measurement within our lab can be most easily reduced by buying a vacuum gauge with a lower uncertainty.
References

